Ma2a Practical – Recitation 7

November 15, 2024

Exercise 1. (Revisit) Let $x(t)$ be a solution of the IVP

$$x'' = 2x - 4x^3, \quad x(0) = 1,\, x'(0) = 0.$$

Is it true that $x(t)$ is a periodic function? Draw the phase diagram of the system

$$\begin{cases} x' = y \\ y' = 2x - 4x^3 \end{cases}$$

Exercise 2. (See Chapter 9.1 Exercise 6 and 19) Consider the following system of D.E.:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

1. Find the eigenvalues of the matrix.
2. The trajectories of the system can be converted into the following equation:

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x - 2y}{2x - 5y}$$

   which is an exact D.E.
3. Solve the above exact D.E.:

   $$x^2 - 4xy + 5y^2 = C$$

   where C is a constant. Conclude that the phase portrait is a family of ellipse.

Exercise 3. (See Chapter 9.3 Exercise 7) Consider the following system of D.E.:

$$\frac{dx}{dt} = 1 - y$$

$$\frac{dy}{dt} = x^2 - y^2$$

1. Find all critical points.
2. Near each critical points, find the correspoonding linear systems.
3. Find the eigenvectors of all the linear systems and draw conclusions[1] about the nonlinear system.

Theorem 9.3.2 Let r_1 and r_2 be the eigenvalues of the linear system (1) corresponding to the locally linear system (4). Then the type and stability of the critical point (0, 0) of the linear system (1) and the locally linear system (4) are as shown in Table 9.3.1.

TABLE 9.3.1 Stability and Instability Properties of Linear and Locally Linear Systems

	Linear System		Locally Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

Exercise 4. (See Chapter 9.7 Example 1) In this exercise, we will study the periodic solution of the nonlinear D.E. Now consider the following system of D.E.:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + y - x(x^2 + y^2) \\ -x + y - y(x^2 + y^2) \end{bmatrix}$$

1. Express $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
2. Show that $r = 1$ and $\theta = -\frac{t^2}{2} + \theta_0$ is a periodic solution of this D.E.
3. Find the general solution.
4. Study the stability of this periodic solution.

[1] see theorem 9.3.2 in textbook

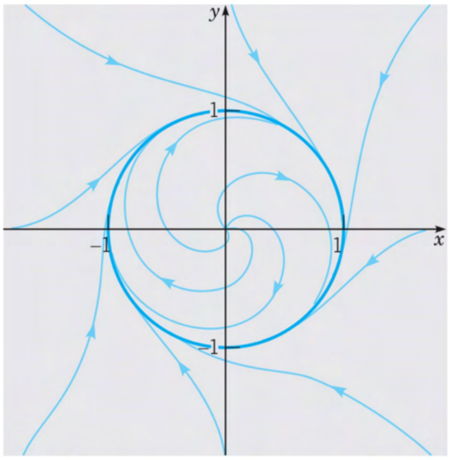


FIGURE 9.7.1 Trajectories of the system (4); the circle $r = 1$ is a limit cycle.

补充 notes Recitation 7.

Thm(uniquess and existence)

similary for: $\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1(t, x_1, \cdots, x_n) \\ \vdots \\ F_n(t, x_1, \cdots, x_n) \end{pmatrix}$

$y' = f(t, y)$, $y(0) = 0$

if f and $\frac{\partial f}{\partial y}$ are continuous, around $(0,0)$, then $\exists$ (a small interval in t) $0 < t < \varepsilon$ s.t. $y = \phi(t)$ is the uniquen solution.

Corollary: If the trajectory of $y = \phi(t)$ closes up, i.e. $\exists$ t_0 s.t. $y(0) = y(t_0)$ then y is periodic.

proof. consider two solutions $y(t)$ and $y(t+t_0)$.

then at $t=0$, $y(0) = y(t_0)$ $\Rightarrow$ $y(t) = y(t+t_0)$ $\forall t$.

Corollary: given autonomous eq. $\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$. the critical points are $\begin{cases} F = 0 \\ G = 0 \end{cases}$.

then integral curve from non-critical point can't pass critical point.

proof: say $(0,0)$ is critical, then $\begin{cases} x(t) = 0 \\ y(t) = 0 \end{cases}$ is a solution.

if $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ is a solution, s.t. $\begin{cases} x_2(t_0) = 0 \\ y_2(t_0) = 0 \end{cases}$ then $\begin{cases} x_2 = x_1 \\ y_2 = y_1 \end{cases}$ $\Rightarrow$ constant solution contradicts.

Matrix representation of conic sections.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If $(B^2 - 4AC)$ $\begin{cases} > 0 & \text{circle / ellipse} & \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ = 0 & \text{parabola} & y = a(x-h)^2 + b \\ < 0 & \text{hyperbola} & \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \end{cases}$

e.g: $x^2 - xy + y^2 - 3y - 1 = 0$

① change center by translation

$\begin{cases} x = x' + h \\ y = y' + h \end{cases}$ (h, k) are new center

$\rightarrow x'^2 - x'y' + y'^2 + x'(2h-k) + y'(-h+2k-3) + h^2 - hk + k^2 - 3k + 1 = 0$

no 1-st. $\begin{cases} 2h = k \\ -h + 2k - 3 = 0 \end{cases} \Rightarrow \begin{cases} h = 1 \\ k = 2 \end{cases}$ i.e. $x'^2 - x'y' + y'^2 = 4$

② No $x'y'$, by rotation.

$x' = X\cos\theta - Y\sin\theta$

$y' = X\sin\theta + Y\cos\theta$

$\rightarrow XY(\sin^2\theta - \cos^2\theta) + X^2(\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta) + Y^2(\cos^2\theta + \sin^2\theta + \sin\theta\cos\theta) = 4$

s. $\theta = \pm\frac{\pi}{4}$

$\rightarrow \frac{X^2}{8} + \frac{3Y^2}{8} = 1$

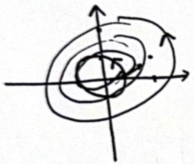
$\begin{cases} x = \frac{\sqrt{2}}{2}(X - Y) + 1 \\ y = \frac{\sqrt{2}}{2}(X - Y) + 2 \end{cases}$

Recitation:

Exer1: $\det\begin{pmatrix} 2-x & -5 \\ 1 & -2-x \end{pmatrix} = (x-2)(x+2)+5 = x^2-4+5 = x^2+1$

① $x = \pm i$

$\begin{cases} r = c \text{ constant} \\ \theta = -\mu t + \theta_0 = \end{cases}$



$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

法一: $x' = A^{-1}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}A^{+}x$

$\vec{y}' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\vec{y}.$ $\quad y = A\vec{x}.$

$\Rightarrow$ solve this

$\begin{cases} r = c \\ \theta = -\mu t + \theta_0 \end{cases}$

$x = A^{-1}y.$

法二 = ② $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x-2y}{2x-5y}$

So $(2x-5y)dy + (x+2y)\,dx = 0$

since $2 = 2$ exact.

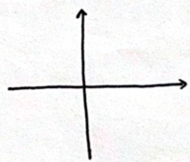
$\begin{cases} \partial_y \psi = 2x - 5y \\ \partial_x \psi = -x + 2y \end{cases} \Rightarrow \begin{cases} \psi = 2xy - \frac{5}{2}y^2 + f(x) \\ 2y + f'(x) = -x + 2y \end{cases}$

$\Rightarrow f'(x) = -x$

$f(x) = -\frac{x^2}{2} + c$

So $\psi = 2xy - \frac{5}{2}y^2 + \left(\frac{-x^2}{2} + c\right) = 0$

$x^2 + 5y^2 - 4xy = c$



Exer2:

$x = X\cos\theta - Y\sin\theta$

$y = X\sin\theta + Y\cos\theta$

$\Rightarrow X^2\cos^2\theta + Y^2\sin^2\theta - \cancel{2XY\sin\theta\cos\theta}$

$+5(X^2\sin^2\theta + Y^2\cos^2\theta + \cancel{2XY\sin\theta\cos\theta})$

$-4(X^2\sin\theta\cos\theta - Y^2\sin\theta\cos\theta - \cancel{XY\sin^2\theta + XY\cos^2\theta}) = c$

then $-2\sin\cos + 10\sin\cos + 4\sin^2 - 4\cos^2 = 0$

$\sin^2 + 2\sin\cos - \cos^2 = 0 \qquad \sin 2\theta = \cos 2\theta$

$2\theta = \frac{\pi}{4}$ e.g. $\theta = \frac{\pi}{8}$

$\rightsquigarrow (3-2\sqrt{2})x^2 + (3+2\sqrt{2})y^2 = c.$

Exer2:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ x^2-y^2 \end{pmatrix}$$

See textbook P522

for how to find A using Jacobian (equivalently)

① critical points:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1-y \\ x^2-y^2 \end{pmatrix} = 0 \quad \text{so} \quad \begin{pmatrix} x=1 \\ y=1 \end{pmatrix} \quad \begin{pmatrix} x=-1 \\ y=1 \end{pmatrix}$$

② (A) • let $\vec{u} = \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \sim 0$ then

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 1-(1+u_2) \\ (u_1+1)^2-(u_2+1)^2 \end{pmatrix} = \begin{pmatrix} -u_2 \\ u_1^2-u_2^2+2u_1-2u_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u_1^2-u_2^2 \end{pmatrix}.$$

✓ since $g_2(u_1,u_2) = u_1^2-u_2^2$ twice differentiable. then it's locally linear.

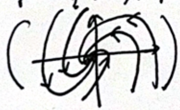
(B) • let $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \vec{u} = \begin{pmatrix} x+1 \\ y-1 \end{pmatrix}$. then

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 1-(u_2+1) \\ (u_1-1)^2-(u_2+1)^2 \end{pmatrix} = \begin{pmatrix} -u_2 \\ u_1^2-u_2^2-2u_1-2u_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u_1^2-u_2^2 \end{pmatrix}$$

③ (A) $A = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}$: $\lambda_1 = -1-i$, $\lambda_2 = -1+i$; $v_1 = \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{1+i}{2} \\ 1 \end{pmatrix}$

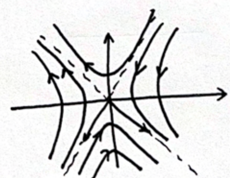
$A(\vec{v}_1, \vec{v}_2) = (\vec{v}_1, \vec{v}_2) D$.

- $-1<0$. S.P: spiral point, asymptotically stable. $\Rightarrow$ NL: spiral & asymp stable.



(B) $A = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix}$: $\lambda_1 = -\sqrt{3}-1 < 0$, $\lambda_2 = \sqrt{3}-1 > 0$; $v_1 = \begin{pmatrix} \frac{\sqrt{3}-1}{2} \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{-\sqrt{3}-1}{2} \\ 1 \end{pmatrix}$

- saddle point unstable $\Rightarrow$ saddle point unstable!

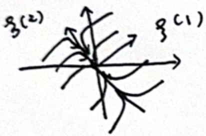


$\dot{x} = A\vec{x}$

Case 1: real, unequal eigenvalues of same sign

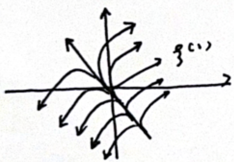
① $\vec{x} = c_1 \cdot \xi^{(1)} \cdot e^{r_1 t} + c_2 \xi^{(2)} \cdot e^{r_2 t} = \cdot e^{r_2 t} \left(c_2 \cdot \xi^{(2)} \cdot \cancel{e^{r_2 t}} + c_1 \xi^{(1)} \cdot e^{(r_1 - r_2)t} \right)$

- if $r_1 < r_2 < 0$,



node / nodal sink

② if $0 < r_2 < r_1$, then same but reverse

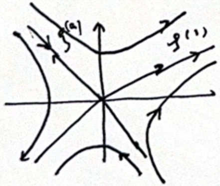


node / nodal source

Case 2: real, unequal eigenvalues of opposite signs.

$\vec{x} = c_1 \cdot \xi^{(1)} e^{r_1 t} + c_2 \cdot \xi^{(2)} \cdot e^{r_2 t}$

$r_1 > 0$, $r_2 < 0$.



Saddpoint

Case 3: Equal eigenvalues, $r_1 = r_2 = r$.

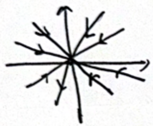
① two independent eigenvectors.

proper node   star point

$$\vec{x} = c_1 \cdot \xi^{(1)} e^{rt} + c_2 \cdot \xi^{(2)} \cdot e^{rt}$$

$$= (c_1 \xi^{(1)} + c_2 \xi^{(2)}) e^{rt}$$

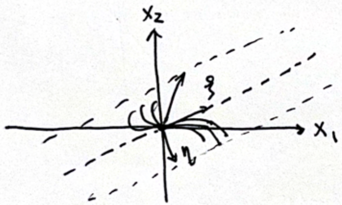
$r<0$



② one independent vector ξ

$$\vec{x} = c_1 \cdot \xi e^{rt} + c_2 (\xi \cdot t \cdot e^{rt} + \eta \cdot e^{rt}) = (c_1 \cdot \xi + c_2 \cdot \xi t + c_2 \eta) e^{rt}$$

where ξ is eigenvector, η is generalized eigenvector for the repeated eigenvalue.



$r<0$

(1) large t, $c_2 \cdot \xi \cdot te^{rt}$ dominants $\Rightarrow$ to 0 & tangent to ξ.

Case 4: Complex eigenvalues

① $\lambda \pm i\mu$ s.t. $\lambda \neq 0$

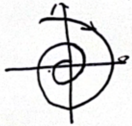
consider $\vec{X}' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \vec{x}$ <u>Problem 22</u> <u>spiral point</u>

$\rightsquigarrow \begin{cases} r = c \cdot e^{\lambda t} \\ \theta = -\mu t + \theta_0 \end{cases}$, θ_0 is value of θ at $t=0$, $\tan\theta_0 = \frac{x_2(0)}{x_1(0)}$

(1) if $\mu > 0$, θ decreases

(2) $t \to \infty$, $r \to 0$ if $\lambda < 0$

$r \to \infty$ if $\lambda > 0$



② $\lambda = 0$, $\pm i\mu$

$\vec{X}' = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \mu$ <u>general</u>: ellipse at 0

$\to \begin{cases} r = c \\ \theta = -\mu t + \theta_0 \end{cases}$

so circle